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RELATIONS AMONG THE MULTIPLIERS FOR PROBLEMS WITH BOUNDED STATE CONSTRAINTS

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In previous articles, the author established certain necessary conditions for control problems with constraints of the form $\psi^\alpha(t,x)\leq 0 \quad \alpha=1,\ldots,m$. These conditions involve certain multiplier functions $\mu_\alpha(t)$ of the derivatives of the above constraints together with multiplier constants $K^\alpha$ used in the transversality relation. In this note, it is shown that these terms satisfy $\mu_\alpha(t^0) \leq K^\alpha$ with $\mu_\alpha(t^0) = K^\alpha$ if $\psi^\alpha(t^0) < 0$.

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## 1. Introduction

We consider the following problem. Let A be the class of arcs $a$ :

$$a: \qquad x^i(t) \qquad u^k(t) \qquad b^\sigma \qquad t^0 \leq t \leq t^1$$
$$i=1,\ldots,N \qquad k=1,\ldots,K \qquad \sigma=1,\ldots,r$$

which[1] have points $t, x(t), u(t)$ in a region R in t-x-u space, b in a region B in b space and $u(t)$ piecewise continuous, and which satisfy the conditions

(1-1) $\qquad \dot{x}^i(t) = f^i(t,x(t),u(t)) \qquad i=1,\ldots,N$

(1-2) $\qquad \psi^\alpha(t,x(t)) \leq 0 \qquad \alpha=1,\ldots,m$

(1-3) $\qquad I_\gamma(a) \leq 0 \quad 1 \leq \gamma \leq p' \qquad I_\gamma(a) = 0 \quad p' < \gamma < p$

(1-4) $\qquad x^i(t^s) = X^{is}(b) \qquad s=0,1 \qquad 1 \leq i \leq N$

where:

$$I_\gamma(a) = g_\gamma(b) + \int_{t^0}^{t^1} L_\gamma(t,x(t),u(t))dt \qquad \gamma=1,\ldots,p \quad .$$

It is desired to minimize the functional

(1-5) $$I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x(t),u(t))dt$$

on the class A .

The functions $\psi^\alpha$ are assumed to be of class $C^2$ on R while the functions $f^i$ , $L_\gamma$ , $g_\gamma$ , $X^{is}$ are of class $C^1$ on R or B as the case may be.

Assume, next, that the arc

$$a_0: \qquad x_0(t) \qquad u_0(t) \qquad b_0 \qquad t^0 \leq t \leq t^1$$

---

[1]Unless otherwise specified, the symbols $i,k,\sigma,\alpha$ will have the respective ranges $1 \leq i \leq N$ , $1 \leq k \leq K$ , $1 \leq \sigma \leq r$ , $1 \leq \alpha \leq m$ .

is a solution to our problem and define the functions[2]

(2) $$\phi^\alpha(t,x,u) = \psi^\alpha_t + \psi^\alpha_{x^i} f^i \qquad \alpha=1,\ldots,m \ .$$

For arcs in the class A , these functions act as $d\psi^\alpha/dt$ along these arcs. We assume that the matrix

(3) $$\begin{bmatrix} \phi^\alpha_{u^k} & \delta_{\alpha\beta}\psi^\beta \end{bmatrix} \qquad \alpha,\beta=1,\ldots,m$$

(where $\delta_{\alpha\beta}$ is the Kronecker delta) has rank $m$ on the set $R_0$ of points $(t,x_0(t),u)$ satisfying

(4) $$\psi^\alpha \leq 0$$
$$\phi^\alpha \geq 0 \text{ for all } \alpha \text{ with } \psi^\alpha = 0 \text{ or } \phi^\alpha \leq 0 \text{ for all } \alpha \text{ with } \psi^\alpha = 0$$
$$1 \leq \alpha \leq m \ .$$

Referring to Theorem 3.1 of [1] and to the quantities $\mu_\alpha(t)$ , $K^\alpha$ of that theorem, we prove[3] the following result:

Lemma: For each $\alpha$ we have

(5) $$\mu_\alpha(t^0) \leq K^\alpha \text{ with } \mu_\alpha(t^0) = K^\alpha \quad \text{if} \quad \psi^\alpha(t^0) < 0 \ .$$

## 2. Proof of the Lemma

It is convenient to prove this result by first transforming the problem. In section 4 of [1] the problem stated above is shown to be equivalent to a

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[2] For a function $M(t,x,u,b)$ , the symbols $M_t$ , $M_{x^i}$ , $M_{u^k}$ , $M_{b^\sigma}$ will denote first partial derivatives with respect to the indicated variable. Also, unless otherwise noted, repeated indices are summed. Finally for a function $K(t,x,u)$ evaluated on the arc $a_0$ as $K(t,x_0(t),u_0(t))$, we shall write $K(t)$.

[3] In Theorem 3.2 of [1], the multipliers $\mu_\alpha(t)$ are modified (by the addition of additive constants) from those of Theorem 3.1 of [1] . The results of this note then imply associated results to the multipliers of that theorem. Similar remarks hold in the Theorems of [2] .

reformulated problem (with superscript bars used on quantities in the reformulated problem to distinguish them from the original problem so that for example, $\bar{\psi}^\alpha$ replaces $\psi^\alpha$ )[4] with functions $\bar{\psi}^\alpha$ , $\bar{\phi}^\alpha$ formed from the functions $\psi^\alpha$ , $\phi^\alpha$ and with the major distinction from the above problem being that the assumption involving (3) is replaced by the statement that the matrix

$$(6) \qquad \left[ \bar{\phi}^\alpha_{u^k} \right]$$

has rank m at points in $\bar{D}$ . Here $\bar{D}$ is the set of points $(t,\bar{x}_0(t),u)$ in $\bar{R}_0$ with $u = \bar{u}_0(t)$ or for arbitrary u with t interior to an interval of continuity of $\bar{u}_0(t)$ . Now $\bar{\phi}^\alpha = \frac{d\bar{\psi}^\alpha}{dt}$ and so (6) implies in particular that

$$(7) \qquad \left[ \bar{\psi}^\alpha_{x^i}(t^0) \right] \qquad \text{has rank } m .$$

The theorem for this latter problem is Theorem 6.1 of [1] and as shown in section 7 of [1], the terms $\mu_\alpha(t)$, $K^\alpha$ of that theorem and of Theorem 3.1 of [1] for the original problems are the same. In addition, $\bar{\psi}^\alpha(t^0) = 0$ iff $\psi^\alpha(t^0) = 0$ $\alpha=1,\ldots,m$ , as shown in (36) of [1]. This proving our lemma for the reformulated problem will prove it also for the original problem.

We concentrate on the reformulated problem of section 4 of [1].

In order now to prove the first inequality of (5), assume that $\eta$ is an index such that

$$(8) \qquad \bar{\psi}^\eta(t^0) < 0$$

and let h be any N dimensional vector such that $\bar{\psi}^\eta_{x^i}(t^0)h^i \neq 0$ . Now,

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[4] Also the dimensions of the variables x,u,b, change in the reformulated problem, however we shall not go into that here.

according to (7), we can select a vector $d$ such that

$$\bar{\psi}^{\eta}_{x^i}(t^0)d^i = \bar{\psi}^{\eta}_{x^i}(t^0)h^i \tag{9}$$

$$\bar{\psi}^{\alpha}_{x^i}(t^0)d^i = 0 \qquad \alpha \neq \eta \quad .$$

Next, select a constant $\delta > 0$ so small that

(10) $$\bar{\psi}^{\eta}(t) < 0 \qquad t^0 \leq t \leq t^0 + \delta$$

and define the $K$ dimensional arc $w$ such that

(11-1)
$$w^{\alpha}(t^0) = \bar{\psi}^{\alpha}_{x^i}(t^0)d^i$$
$$\dot{w}^{\alpha}(t) = \begin{cases} (-\bar{\psi}^{\alpha}_{x^i}(t^0)d^i)\,\frac{2}{\delta} & t^0 \leq t \leq t^0 + \frac{\delta}{2} \\ 0 & t^0 + \frac{\delta}{2} \leq t \leq t^1 \end{cases} \qquad \alpha=1,\ldots,m$$

(11-2) $$w^{\Gamma}(t) \equiv 0 \qquad \Gamma = m+1,\ldots,K \qquad t^0 \leq t \leq t^1 \quad .$$

Then $\omega$ is in the class $W$ of section 13 of [1] and by Lemma 13.1 of [1], we can find an admissible variation[5]

(12) $\delta a$ : $\quad \delta x(t) \quad \delta u(t), \quad \delta b \quad t^0 \leq t \leq t^1$

satisfying

(13-1) $$\delta x^{j_s}(t^0) = d^{j_s} \qquad j_s \neq i_\rho \qquad s=1,\ldots,N-m \ .$$

(13-2) $$\delta b = 0$$

---

[5]See Section 11 of [1].

where $i_\rho$ are the indices of (108) of [1] and also satisfying

$$\bar{\psi}^\alpha_{x^i}(t)\delta x^i(t) = \delta\bar{\psi}^\alpha(t) = w^\alpha(t) \qquad \alpha=1,\ldots,m \tag{14}$$
$$\delta\bar{\phi}^\Gamma(t) = w^\Gamma(t) \qquad \Gamma=m+1,\ldots,K \qquad t^0 \le t \le t^1$$

where: $\delta\bar{\psi}^\alpha(t)$ , $\delta\bar{\phi}^\Gamma(t)$ indicate[6] the variations in these quantities due to the variation $\delta a$ and where $\bar{\phi}^\Gamma$ are the functions of section 8 of [1]. According to the above and by the admissibility of $\delta a$ , we have that

$$\delta\bar{\phi}^\alpha(t) = \frac{d}{dt}\,\delta\bar{\psi}^\alpha(t) = \dot{w}^\alpha(t) = \begin{cases} \left(-\bar{\psi}^\alpha_{x^i}(t^0)d^i\right)\frac{2}{\delta} & [t^0\ ,\ t^0+\frac{\delta}{2}\,] \\ 0 & [t^0+\frac{\delta}{2}\ ,\ t^1\,] \end{cases} \tag{15}$$

and by (14) and (11-2) also

$$\delta\bar{\phi}^\Gamma(t) \equiv 0 \qquad \Gamma = m+1,\ldots,K \qquad t^0 \le t \le t^1 \quad . \tag{16}$$

In addition, by (11-1), (13-1), and (14) evaluated at $t = t^0$ , we have

$$\bar{\psi}^\alpha_{x^{i_\rho}}(t^0)[d^{i_\rho} - \delta x^{i_\rho}(t^0)] = 0 \qquad \rho,\alpha = 1,\ldots,m \tag{17}$$

where $i_\rho$ are the indices of (108) of [1]. Then by the nonsingularity of the matrix $\left[\bar{\psi}^\alpha_{x^{i_\rho}}(t^0)\right]$ (see (108) of [1]), we see that $\delta x^{i_\rho}(t^0) = d^{i_\rho}$ $\rho=1,\ldots,m$ , so that together with (13-1) we obtain

$$\delta x^j(t^0) = d^j \qquad j=1,\ldots,N \quad . \tag{18}$$

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[6] See section 11 of [1] .

Next, by (155-2) and Lemmas 11.1 and 15.1 all of [1], together with (15), (16) and (18), we get by computing the variation of the functionals introduced in (69) and (70) of [1] that

$$(19)\quad \tilde{\lambda}_\rho \int_{t^0}^{t^0+\delta/2} F_{\rho u^k}\, \zeta_\alpha^k (-\bar{\psi}^\alpha_{x^i}(t^0)d^i)\, \frac{2}{\delta}\, dt - \tilde{\lambda}_{p+N+i} d^i \geq 0 \qquad \rho=0,1,\ldots,p+N$$

where $F_\rho$, $\zeta_\alpha^k$, $\tilde{\lambda}_{p+N+i}$ are quantities introduced in section 8 of [1].

Using the relations (76-1) of [1] (between $\tilde{\lambda}_{p+N+i}$ and $K^\alpha$) and (9), we see that (19) becomes

$$(20)\quad \tilde{\lambda}_\rho \int_{t^0}^{t^0+\delta/2} F_{\rho u^k}\, \zeta_\eta^k (-\bar{\psi}^\eta_{x^i}(t^0)h^i)\, \frac{2}{\delta}\, dt - K^\eta \bar{\psi}^\eta_{x^i}(t^0)h^i \geq 0 \quad (\eta \text{ not summed})$$

where $K^\eta$ is that term referred to in our present lemma which is associated with $\bar{\psi}^\eta$. Furthermore, by the definition of $\mu_\alpha(t)$ in (74) and (76) of [1] then (20) is:

$$(21)\quad \left(\bar{\psi}^\eta_{x^i}(t^0)h^i\right)\left[\frac{2}{\delta}\int_{t^0}^{t^0+\delta/2} \mu_\eta(t)\,dt - K^\eta\right] \geq 0 \qquad (\eta \text{ not summed}).$$

According to the properties of the multipliers $\mu_\alpha(t)$, we can by reducing $\delta$ if necessary, guarantee that $\mu_\eta(t)$ is continuous on $[t^0, t^0+\delta/2]$. Then by taking the limit of the expression in (21), we get that

$$(22\text{-}1)\qquad \bar{\psi}^\eta_{x^i}(t^0)h^i[\mu_\eta(t^0) - K^\eta] \geq 0 \qquad (\eta \text{ not summed}).$$

Now we can repeat this same construction with $-h$ replacing $h$ and so get

$$(22\text{-}2)\qquad \bar{\psi}^\eta_{x^i}(t^0)(-h^i)[\mu_\eta(t^0) - K^\eta] \geq 0 \qquad (\eta \text{ not summed}).$$

Thus, (22) implies that for any vector $h$ with $\bar{\psi}^{\eta}_{x^i}(t^0)h^i \neq 0$ , then

(23) $$\bar{\psi}^{\eta}_{x^i}(t^0)h^i[\mu_{\eta}(t^0) - K^{\eta}] = 0 \qquad (\eta \text{ not summed})$$

which implies that

(24) $$\mu_{\eta}(t^0) = K^{\eta} \quad .$$

Since $\bar{\psi}^{\eta}$ was an arbitrary constraint such that $\bar{\psi}^{\eta}(t^0) < 0$ , then the second statement of our lemma is proven.

In order to prove the first statement of our lemma, let $\eta$ be an index such that

(25) $$\bar{\psi}^{\eta}(t^0) = 0$$

and let $h$ be a vector such that

(26) $$\bar{\psi}^{\eta}_{x^i}(t^0)h^i \leq 0 \quad .$$

Then as above, pick a vector $d$ such that (9) is true and define the arc $w$ as in (11) where $\delta$ is selected so that the multiplier $\mu_{\eta}(t)$ is continuous on $[t^0, t + \delta/2]$. The construction follows identical steps to the above to yield (22-1) while together with (26) and the arbitrariness of $\eta$ , proves the first statement of our lemma and hence also the lemma.

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